Brownian motion and Stochastic Calculus
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Assignment 0—solutions

Exercise 1

Let X_i, $i \in \mathbb{N}$ be i.i.d. with $X_i \sim \mathcal{N}(0,\sigma^2)$ for some $\sigma^2 > 0$. Let us define $Y_0 = 1$ and

$$Y_n = \exp\left(\sum_{i=1}^n X_i - n\frac{\sigma^2}{2}\right), \quad n \in \mathbb{N}.$$

1) Show that $(Y_n)_n$ is a martingale.
2) Snow that $Y_n \longrightarrow 0$, $\mathbb{P}-$a.s. for $n \to \infty$.
3) Is the process $(Y_n)_n$ uniformly integrable? Why/why not?
4) Assume now that $X_i \sim \mathcal{N}(\mu,\sigma^2)$ for some $\mu \in \mathbb{R}$. For which values of $(\mu,\sigma^2) \in \mathbb{R}\times(0,\infty)$ does point 2) still hold?

1) $(Y_n)_n$ **is clearly adapted and integrable since** X_i, $i \in \mathbb{N}$, **are normally distributed. We verify the martingale property. Let us fix** $n \in \mathbb{N}$, $n \geq 2$ **:**

$$\mathbb{E}\left[Y_{n+1}\middle|\mathcal{F}_n^Y\right] = \mathbb{E}\left[\exp\left(\sum_{i=1}^n X_i - (n+1)\frac{\sigma^2}{2}\right)\exp\left(X_{n+1}\right)\middle|\mathcal{F}_n^Y\right] = \exp\left(\sum_{i=1}^n X_i - (n+1)\frac{\sigma^2}{2}\right)\mathbb{E}\left[\exp\left(X_{n+1}\right)\middle|\mathcal{F}_n^Y\right]$$
$$= \exp\left(\sum_{i=1}^n X_i - (n+1)\frac{\sigma^2}{2}\right)\mathbb{E}\exp\left(X_{n+1}\right) = \exp\left(\sum_{i=1}^n X_i - (n+1)\frac{\sigma^2}{2}\right)\exp\left(\frac{\sigma^2}{2}\right) = Y_n, \quad \mathbb{P}-\textbf{a.s..}$$

The case $n = 1$ **can be verified analogously.**

2) **We have**

$$Y_n = \exp\left(\sum_{i=1}^n X_i - n\frac{\sigma^2}{2}\right) = \exp\left(n\left[\frac{1}{n}\sum_{i=1}^n X_i - \frac{\sigma^2}{2}\right]\right), \quad n \in \mathbb{N}.$$

The SLLN yields $\frac{1}{n}\sum_{i=1}^n X_i \longrightarrow 0$ $\mathbb{P}$**-a.s.. The result then follows immediately.**

3) **Uniform integrability and convergence** $\mathbb{P}$**-almost surely to** 0 **would imply convergence in** $\mathbb{L}^1$ **to** 0. **However, this does not hold true since** $\mathbb{E}|Y_n| = 1$ **for any** $n \in \mathbb{N}\cup\{0\}$.

4) **We can write**

$$Y_n = \exp\left(\sum_{i=1}^n X_i - n\frac{\sigma^2}{2}\right) = \exp\left(n\left[\frac{1}{n}\sum_{i=1}^n (X_i-\mu) - \left(\frac{\sigma^2}{2} - \mu\right)\right]\right), \quad n \in \mathbb{N}.$$

Using the same argument, we see that $Y_n \longrightarrow 0$, $\mathbb{P}-$**a.s. holds if** $\mu < \sigma^2/2$.

Exercise 2

Let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space with a (discrete) filtration $\mathbb{F} = (\mathcal{F}_n, n \in \mathbb{N}\cup\{0\})$. Let τ_n be a stopping time for every $n \in \mathbb{N}$. Which of the following are always stopping times?

1) $\sup_{n\in\mathbb{N}} \tau_n$.

2) $\inf_{n\in\mathbb{N}} \tau_n$.

We have

$$\left\{\sup_{n\in\mathbb{N}} \tau_n \le m\right\} = \bigcap_{n\in\mathbb{N}} \{\tau_n \le m\} \in \mathcal{F}_m, \quad m \in \mathbb{N}.$$

Similarly,

$$\left\{\inf_{n\in\mathbb{N}} \tau_n \le m\right\} = \bigcup_{n\in\mathbb{N}} \{\tau_n \le m\} \in \mathcal{F}_m, \quad m \in \mathbb{N}.$$

Hence, both variables are stopping times.

Let us emphasize that the infimum of stopping times is, in general, NOT a stopping time in continuous-time setting.

Exercise 3

Let X_i, $i \in \mathbb{N}$ be i.i.d. random variables with $\mathbb{P}(X_i = 1) = \mathbb{P}(X_i = -1) = 1/2$. Let us set $S_0 = 0$ and

$$S_n = \sum_{i=1}^{n} X_i, \quad n \in \mathbb{N}.$$

(i.e., $(S_n)_n$ is a simple symmetric random walk.) Let us have two finite constants $A \in \mathbb{Z}, A < 0$ and $B \in \mathbb{Z}, B > 0$ and let us set

$$\tau_{S,[B,\infty)} = \inf \{n \in \mathbb{N} : S_n \ge B\},$$
$$\tau_{S,(A,B)^C} = \inf \{n \in \mathbb{N} : S_n \notin (A, B)\}.$$

You may assume (without having to prove it) that $\tau_{S,[B,\infty)} < \infty$ and $\tau_{S,(A,B)^C} < \infty$ $\mathbb{P}$-a.s..

1) Show that it *doesn't* hold $\mathbb{E}S_0 = \mathbb{E}S_{\tau_{S,[B,\infty)}}$.

2) Recall the statement of optional sampling theorem.

3) Why can't we use optional sampling theorem for point 1.)?

4) Show that $0 = \mathbb{E}S_0 = \mathbb{E}S_{\tau_{S,(A,B)^C}}$.

5) Compute $\mathbb{P}(S_{\tau_{S,(A,B)^C}} = A)$ and $\mathbb{P}(S_{\tau_{S,(A,B)^C}} = B)$.

1) **We clearly have that** $\mathbb{E}S_0 = 0$ **and, since** $\tau_{S,[B,\infty)} < \infty$**, we have** $S_{\tau_{S,[B,\infty)}} = B$, $\mathbb{P}$**-a.s.. Therefore,** $\mathbb{E}S_{\tau_{S,[B,\infty)}} = B \neq 0 = \mathbb{E}S_0$.

2) **and** 3) : **It is clear that the assumptions of optional sampling theorem are never satisfied. The process** $(S_n)_n$ **is not UI and the stopping time** $\tau_{S,[B,\infty)}$ **(though being finite** $\mathbb{P}$**-a.s.) is not bounded.**

4) **The stopped process** $(S_{n\wedge\tau_{S,(A,B)^C}})_n$ **is bounded and hence uniformly integrable. It follows from optional stopping theorem that**

$$0 = \mathbb{E}S_0 = \mathbb{E}S_{n\wedge\tau_{S,(A,B)^C}}, \quad n \in \mathbb{N}.$$

Passing to the limit as $n \to \infty$ **and using the DCT (exploiting again that** $(S_{n\wedge\tau_{S,(A,B)^C}})_n$ **is bounded) yields**

$$0 = \mathbb{E}S_0 = \mathbb{E}S\tau_{S,(A,B)^C}.$$

5) **The previous gives (recall that $\tau_{S,(A,B)^C} < \infty$, $\mathbb{P}-$a.s.)**

$$0 = \mathbb{E}S_0 = \mathbb{E}S\tau_{S,(A,B)^C} = A \cdot \mathbb{P}(S_{\tau_{S,(A,B)^C}} = A) + B \cdot \left[1 - \mathbb{P}(S_{\tau_{S,(A,B)^C}} = A)\right].$$

It follows that

$$\mathbb{P}(S_{\tau_{S,(A,B)^C}} = A) = \frac{B}{B-A}$$

and

$$\mathbb{P}(S_{\tau_{S,(A,B)^C}} = B) = 1 - \mathbb{P}(S_{\tau_{S,(A,B)^C}} = A) = \frac{A}{A-B}.$$

Exercise 4

Recall martingale convergence theorems and martingale inequalities.